Categoricity and Open-Ended Axiom Systems

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Intuition and Reason:
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Logic of Mathematical Practice

- This is part of a program (long) in progress to provide a logical framework for mathematical practice.
- *Caveat*: details subject to change.

Practice vs. Logicians' Axiomatics

- Mathematicians pay little or no attention to logic or formal axiomatic systems (e.g., PA or ZF).
- Ubiquity of basic structures as givens: N, Q, R, C, etc.

Practice vs. Logicians' Axiomatics (cont'd)

- Ubiquity of proof by induction and definition by recursion on N
- Ubiquity of the l.u.b. principle for R
- None of these dealt with from logicians' point of view via formal systems in fixed vocabularies

Open-Ended Mathematical Practice (OEMP)

- Treat basic schemata in logic, arithmetic, analysis and set theory in an open-ended way:
- $P0 \wedge \forall n[Pn \rightarrow P(n+1)] \rightarrow \forall n(Pn)$
- $\sup\{x \in R: Px\}$ in R for P bounded
- $\{x \in a: Px\}$ is a set
- Which P?

Basic Features of OEMP

- Ontology is determinately pluralistic: Heterogeneous universe U of sets, functions, operations, classes, properties, etc.
- Sets and functions *extensional*; operations, classes and properties *intensional*.

Basic Features of OEMP (cont'd)

- Operations applicable across U
- Sets include N, R, function sets $A \to B$, power sets $\wp(A)$, etc.
- Fundamental schemata for these are open-ended

Universal Operational Framework

- Objects: a, b, c,..., x, y, z range over U
- Pairs, Tuples: (x, y), iterate for tuples
- Operations: f, g, h,... intensional objects in U, given by rules; possibly partial; $fx\downarrow$ for "f is defined at x"; fxy or f(x, y) for binary operations

Operational Axioms

- Either untyped partial lambda-calculus or partial combinatory algebra (Curry combinators), augmented by pairing and projection operators and definition by cases.
- $s \simeq t$ means: if either $s\downarrow$ or $t\downarrow$ then both are defined and $s = t$.

General Recursor

- <u>Theorem</u> There is a term r such that for all f, rf$\downarrow$ and rfx $\simeq$ f(rf)x, i.e. for g = rf, we have gx $\simeq$ fgx for all x.

Arithmetic

- (N, Sc, Pd, 0) is assumed to satisfy the usual axioms for 0, successor (Sc) and predecessor (Pd) and the open-ended scheme of induction.
- The recursion theorem implies primitive recursion on N into U, using a positive QF applicative property P.

Categoricity of N

- Given $(N', Sc', Pd', 0')$ satisfying the axioms of the N structure, define g by $g(0) = 0'$ and $g(Sc(x)) = Sc'(g(x))$ for each x in N; similarly for g'.
- Prove $g: N \rightarrow N'$ and $g'(g(x)) = x$ for each x in N, by induction on N; hence one-one.

Categoricity of N (cont'd)

- Similarly, prove $g(g'(y)) = y$ for all y in N' by induction on N'; hence g is onto.
- Thus $(N, Sc, Pd, 0) \cong (N', Sc', Pd', 0')$
- The principle of charity.

Categoricity by PRA

- This part of OEMP can be interpreted in Sigma_1-IA, hence its strength is bounded by PRA by Parsons' and Mints' theorem.
- Simpson and Yokoyama (APAL 2012) show N-categoricity equivalent to WKL_0 in 2nd order arithmetic over RCA_0 but proof more complicated.

Higher Order Categoricity and OEMP

- By addition of suitable set and ordinal construction operators, obtain categoricity of power set and its finite and transfinite iteration.
- OEPM is consistent relative to OST + Pow, and thence to KP + Pow.

The End