WHY ISN'T THE CONTINUUM PROBLEM ON THE MILLENNIUM ($1,000,000) PRIZE LIST?

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CSLI Workshop on Logic, Rationality and Intelligent Interaction

Stanford, June 1, 2013

Why isn't the Continuum Problem on the Millennium Prize list?

- The tip of an iceberg of questions as to what counts as mathematics.
- The same questions have occured at many points in the historical evolution of mathematics.

What Counts as Mathematics?
A Nexus of Questions

- What counts as a mathematical notion?
- What counts as a mathematical problem?
- What counts as the solution to a mathematical problem?
- <u>Note</u>: as mathematics evolves, what's in can become out, and what's out can become in.

The Millennium Prize List (Y 2000)

- The Millennium Prize List: 7 famous unsolved problems, including the Riemann Hypothesis, Poincaré Conjecture (subsequently solved), P vs NP, etc.
- The prize: $1,000,000 each.
- The criteria for the problems on the list: Should be historic, central, important, and difficult.

Hilbert's [23] Mathematical Problems (Paris ICM 1900)

- Problem 1: Cantor's Continuum Hypothesis (CH)
- "The investigations of Cantor... suggest a very plausible theorem [namely, CH], which in spite of the most strenuous of efforts, no one has succeeded in proving."

Wikipedia lists of Unsolved Problems in Mathematics

- “Unsolved problems in mathematics”(125, of which 9 in set theory)
- “List of conjectures” (115, of which none in set theory)
- None of these includes CH
- What happened in between?

What is the Continuum?

- The geometrical continuum
- The arithmetical continuum
- The set-theoretical continuum

The Arithmetical Continuum (aka the Real Numbers R)

- Measurement numbers on a two-way infinite straight line
- Relative to: an origin, a unit of length, and a positive direction.
- Every point is represented by a real number.

The Arithmetical Continuum (continued)

- 0 represents the origin.
- 1 represents the r. h. end point of the unit interval [0, 1].
- Binary representation: every infinite sequence of 0s and 1s represents a point in [0, 1] (e.g., 01101001...), and vice-versa.

The Set-Theoretical Continuum

- 2^N, the set of all N-termed sequences of 0's and 1's, where N is the set of natural numbers {0, 1,...}.
- Or P(N), the set of all subsets of N.
- R, 2^N, P(N) all in 1-1 correspondence.

“What is Cantor’s Continuum Problem?” (Gödel 1947)

- “Cantor’s continuum problem is simply the question: How many points are there on a straight line in Euclidean space... In other terms: How many different sets of integers do there exist?”
- “The analysis of the phrase ‘how many’ leads unambiguously to a definite meaning for the question.”

Cantor's (1873) Analysis of 'How Many'

- Explain how many elements a set has in terms of when two sets have the same number of elements.
- Card(A) = "the cardinal number of A"
- Card(A) = Card(B) iff A, B can be put in 1-1 correspondence.

Cantor's Analysis of 'How Many' (continued)

- Card(A) < Card(B) iff A is in 1-1 correspondence with a subset of B, but not v.-v.
- Theorem: The Trichotomy Law ⇔

  The Well-Ordering Theorem ⇔

  The Axiom of Choice (AC).

Zermelo's Axiom of Choice (1904)

- Used to prove the Well-Ordering Theorem.
- For many years the subject of much controversy. (Cf. G. H. Moore history)
- Eventually by and large accepted by the mathematical community.
- When is an "axiom" an axiom?

The Continuum is Uncountable

- N = the Natural Numbers,
  R = the Continuum
- <u>Cantor's Theorem</u>: The Continuum is uncountable, i.e. Card(N) < Card(R)
- Equivalently, Card(N) < Card(P(N))
- Proof, by the Diagonal Argument.

Cantor’ Continuum Hypothesis (1878)

- The Continuum Problem: Is there any cardinal number between Card(N) and Card(P(N))?
- The Continuum Hypothesis (CH) says there is no such number.
- The Continuum Problem is simple, natural and basic!

History of Work on CH

- 1878-1930, "Proofs" and "Disproofs"; CH for special kinds of sets; enter GCH
- 1930-1938, Equivalents of CH and GCH
- 1938-1940, Gödel, Consistency of GCH
- 1963, Cohen, Independence of CH
- 1963- : The metamathematics of set theory really takes off!

The Relative Consistency of GCH
Gödel (1938-1940)

- Zermelo-Fraenkel (ZF) axiomatic set theory; ZFC = ZF + AC.
- Theorem (Gödel): ZFC + GCH is consistent, if ZF is consistent.
- Proof: The "constructible sets" (L) form an "inner" model of ZFC + GCH within ZF.

“What is Cantor’s Continuum Problem?” Gödel (1947)

- Asserted that CH is a definite problem.
- Conjectured that CH is false!
- New axioms would be needed to settle it.
- Axioms accepted on intrinsic grounds vs. those accepted on extrinsic grounds.
- Large Cardinal Axioms (LCAs)--both kinds.

The Intrinsic Program
Gödel (1947)

- “Small” LCAs , e.g. those for higher and higher inaccessibles à la Mahlo, ought to be accepted for the same reasons one has accepted ZFC.
- But those are all seen to be true in L, so can’t contradict CH.
- So more would be needed for that.

The Extrinsic Program
Gödel (1947)

"There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline...that quite irrespective of their intrinsic necessity they would have to be assumed in the same sense as any well-established physical theory."

“Large” LCAs

- Measurable cardinals (Ulam, 1929)
- Theorem (Scott, 1961): There are no measurable cardinals in L.
- Then, a hierarchy of “large” LCAs: Ramsey, measurable, Woodin, compact, supercompact, huge, superhuge, etc. (Cf. Kanamori, *The Higher Infinite*).

The Continuum Problem for Classes of Subsets of R in DST

- A subset X of R has the Perfect Set Property (PSP) if it is countable or contains a non-empty perfect subset. If X is uncountable and has the PSP then Card(X) = Card(R).
- Luzin, Suslin: Descriptive Set Theory (DST, definable subsets of the continuum): Borel sets, analytic sets (projections of Borel sets), co-analytic sets, and so on through the projective hierarchy.

The Continuum Problem for Classes of Sets in DST (cont'd)

- Theorem (Suslin1917): Every uncountable analytic set has the PSP.
- Theorem Gödel (1938): In L there exist uncountable co-analytic sets without the PSP.

A Case Study in the Extrinsic Program
--Enter the Axiom of Determinacy--

- For each subset X of the continuum, G(X) is a two-person infinite game which ends with an infinite sequence σ of 0s and 1s. Player 1 wins if σ is in X, otherwise Player 2 wins.
- The Axiom of Determinacy (AD) (Mycielski and Steinhaus 1962) :
  For every set X there is a winning strategy for one of the players in G(X).

A Case Study in the Extrinsic Program
--Consequences of AD--

- Theorem: AD contradicts AC. ("Bad!")
- Theorem: AD implies that every set of reals is Lebesgue measurable, has the Baire property and has the Perfect Set Property. ("The Big Three Properties--Good!")

A Case Study in the Extrinsic Program
--Enter Projective Determinacy--

- Projective Determinacy (PD) is AD restricted to the games G(X) for X in the projective hierarchy.
- Theorem: PD implies that every projective set has the big three properties.

A Case Study in the Extrinsic Program
--A Proof [?]of PD--

- "A proof of projective determinacy"(Martin and Steel 1989, with a strengthening by Woodin).
- Theorem: If there exist infinitely many Woodin cardinals then PD holds.
- Success for the Extrinsic Program?

Which parts of Gödel's Extrinsic Program are met by Martin-Steel?

- "axioms so abundant in their verifiable consequences" (?)
- "shedding so much light upon a whole discipline" (?)
- "they would have to be assumed in the same sense as any well-established physical theory" (?)

A (semi-)Circle of Extrinsic Justification:

Woodin's theorems for AD in L(R)

- 1. If there are infinitely many Woodin cardinals with a measurable cardinal above them then AD holds in L(R).
- 2. If AD holds in L(R) then there is an inner model of "there exist infinitely many Woodin cardinals".
- 3. The big three plus one more "good property" of P(R) imply AD holds in L(R).

But What Hope for the Extrinsic Program to settle CH?

- Theorem (Levy and Solovay 1967): CH is consistent with and independent of all ("small" and "large") LCAs that have been considered to date, provided they are consistent with ZF.
- Proof: By Cohen's method of forcing.

What Prospects for CH?
Woodin's Program: Changing the Logic

- Ω-logic: A new, infinitary logic that cannot be altered by forcing, thus avoiding the Levy-Solovay Theorem.
- Woodin's Strong Ω-conjecture implies:
  the statement that not-CH is true is an Ω-consequence of ZFC + (an Ω-complete axiom A).
- The proof assumes the existence of a proper class of Woodin cardinals.
- But in what sense would that settle CH?

Back to the Millennium Prize list

- Board won't say which problems were considered for inclusion.
- Don't know if it was considered and, if so, whether experts were consulted.
- Can only speculate why not included if considered at all.

The Millennium Prize list (cont'd)

- They might have concluded from the Gödel-Cohen results (or, better, the Levy-Solovay results) that CH is an essentially undecidable proposition.
- Or they might have seen it as a definite problem but that no proposed solution of the sort that is in sight would clearly count as decisive for the mathematical community at large for which to commit $1,000,000.
- Or??

Is CH a Definite Mathematical Problem?

- My view: No; in fact it is essentially indefinite ("inherently vague").
- That is, the concepts of arbitrary set and function as used in its formulation even at the level of P(N) are essentially indefinite.
- For, any attempt to sharpen the concept is at basic odds with the idea of "arbitrary subset of a given [infinite] set."
- Even the search for "Ultimate L".

Martin (1976)

"Those who argue that the concept of set is not sufficiently clear to fix the truth-value of CH have a position which is at present difficult to assail. As long as no new axiom is found which decides CH, their case will continue to grow stronger, and our assertion that the meaning of CH is clear will sound more and more empty."

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The End