IS THE CONTINUUM HYPOTHESIS A DEFINITE MATHEMATICAL PROBLEM?

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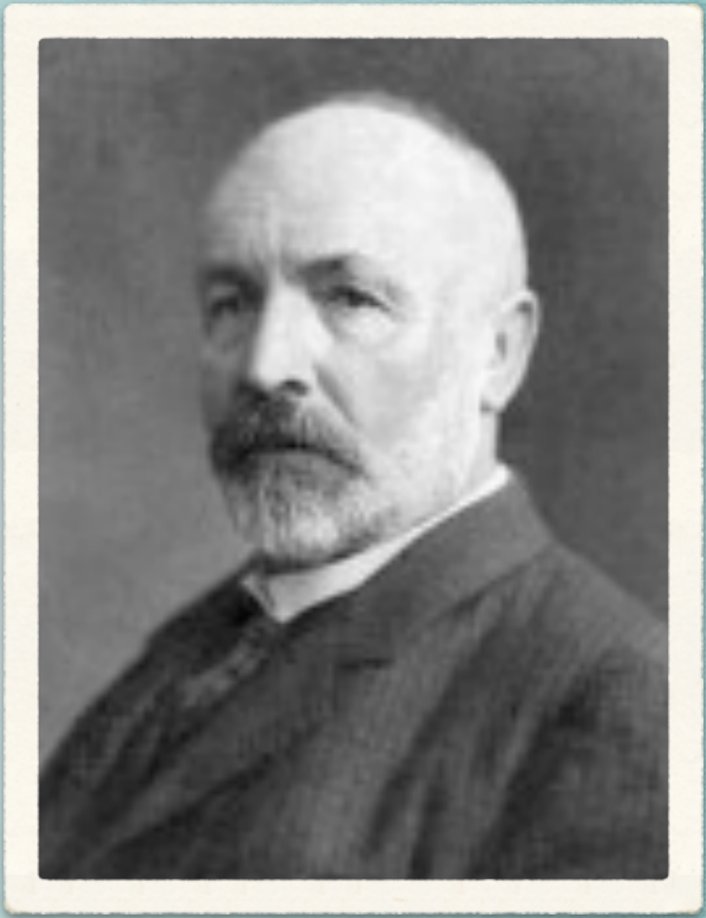
Georg Cantor

1845-1918



What is the Continuum?

- The geometrical continuum
- The arithmetical continuum
- The set-theoretical continuum

The Arithmetical Continuum (aka the Real Numbers)

- Measurement numbers on a two-way infinite straight line
- Relative to: an origin, a unit of length, and a positive direction.
- Every point is represented by a number.

The Arithmetical Continuum
(continued)

- 0 is the number of the origin
- 1 is the number of the right end point of the unit length.
- Binary representation: every infinite sequence of 0s and 1s represents a point in [0, 1] (e.g., 01101001...), and vice-versa.

The Set-Theoretical Continuum

- 2^N, the set of all N-termed sequences of 0s and 1s, where N is the set of natural numbers {0, 1,...}.
- Or $\mathcal{P}(N)$, the set of all subsets of N.
- 2^N, $\mathcal{P}(N)$ and the arithmetical continuum all in 1-1 correspondence.

“What is Cantor’s Continuum Problem?” (Gödel 1947)

“Cantor’s continuum problem is simply the question: How many points are there on a straight line in Euclidean space... In other terms: How many different sets of integers do there exist?

“The analysis of the phrase ‘how many’ leads unambiguously to a definite meaning for the question...”

Cantor's Analysis of 'How Many'

- Two sets A and B have the same number of elements iff they can be put in one-one correspondence.
- Card(A) = the cardinal number of A
- Card(A) = Card(B) iff A, B can be put in one-one correspondence.

Cantor's Analysis of 'How Many' (continued)

- Card(A) < Card(B) iff A is in one-one correspondence with a subset of B, but not Card(A) = Card(B).
- Theorem. The Trichotomy Law ⇔

  The Well-Ordering Theorem ⇔

  The Axiom of Choice (AC).

Countable Sets

- A set is countable if it is either empty or can be enumerated.
- Every finite set is countable
- The countably infinite sets A are just those with Card(A) = Card(N).

The Continuum is Uncountable

- Let C = the Continuum
- Theorem. Card(N) < Card(C)
- Proof, by contradiction: the Diagonal Argument.

Cantor’ Continuum Hypothesis

- The Continuum Problem: Is there any number between Card(N) and Card(C)?
- The Continuum Hypothesis (CH) says that there is no such number.

Hilbert's First Problem (1900)
(HP-4)

- "The investigations of Cantor...suggest a very plausible theorem [namely, CH], which in spite of the most strenuous of efforts, no one has succeeded in proving."
- Hilbert also asked to give an explicit well-ordering of the continuum.

Hilbert's Strange Claim (1925)

- "I would still like to play a last trump. ... The solution of the continuum problem can be carried out by means of the theory [of proofs] I have developed."
- P. Lévy (1964): "Zermelo told me in 1928 that even in Germany nobody understood what Hilbert meant."

The Relative Consistency of CH
Gödel (1938-1940)

- Zermelo-Fraenkel (ZF) axiomatic set theory; ZFC = ZF + AC.
- Theorem. (Gödel) ZFC + CH is consistent, if ZF is consistent.
- Proof: The "constructible sets" (L) form a model of ZFC + CH within ZF.

Hilbert's (Non-)Reaction

- Hilbert had no reaction to Gödel's result. But he was "out of it" by 1937.
- "Memory only confuses thought--I have completely abolished it for a long time. I really don't need to know anything, for there are others, my wife and our maid--they will know." [C. Reid biography]

“What is Cantor’s Continuum Problem?” Gödel (1947)

- Asserted that CH is a definite problem.
- Conjectured that CH is false!
- New axioms would be needed to settle it.
- Axioms accepted on intrinsic grounds vs. those accepted on extrinsic grounds.
- Large Cardinal Axioms (LCAs).

The Intrinsic Program
Gödel (1947)

- According to Gödel, "Small" LCAs , e.g. those for higher and higher inaccessibles à la Mahlo, ought to be accepted for the same reasons one has accepted ZFC.
- But those are all seen to be true in L, so can't contradict CH.

The Extrinsic Program
Gödel (1947)

"There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline...that quite irrespective of their intrinsic necessity they would have to be assumed in the same sense as any well-established physical theory."

"Large" LCAs

- Measurable cardinals (Ulam, 1929)
- Theorem. (Scott 1961) There are no measurable cardinals in L.
- A hierarchy of "large" LCAs: Ramsey, measurable, Woodin, compact, supercompact, huge, superhuge, etc. (Cf. Kanamori, *The Higher Infinite*).

The Continuum Problem for Classes of Sets in DST

- A subset X of the continuum C has the Perfect Set Property (PSP) if it contains a non-empty perfect subset. If X has the PSP then Card(X) = Card(C)
- Luzin and Suslin, Descriptive Set Theory (definable subsets of the continuum): Borel sets, analytic sets (projections of Borel sets), co-analytic sets, and so on through the projective hierarchy.

The Continuum Problem for Classes of Sets in DST (cont'd)

- Theorem (Suslin1917). Every uncountable analytic set has the PSP.
- Theorem Gödel (1938). There exist uncountable co-analytic sets without the PSP in L.

A Case Study in the Extrinsic Program
--Enter the Axiom of Determinacy--

- For each subset X of the continuum, G(X) is a two-person infinite game which ends with an infinite sequence σ of 0s and 1s. Player 1 wins if σ is in X, otherwise Player 2 wins.
- The Axiom of Determinacy (AD) (Mycielski and Steinhaus 1962) :
  For every set X there is a winning strategy for one of the players in G(X).

A Case Study in the Extrinsic Program
--Consequences of AD--

- Theorem. AD contradicts AC. ("Bad")
- Theorem (Mycielski, et al., 1964).
  AD implies that every set of reals is Lebesgue measurable, has the Baire property and has the PSP. ("Good")

A Case Study in the Extrinsic Program
--Enter Projective Determinacy--

- Projective Determinacy (PD) is AD restricted to the games G(X) for X in the projective hierarchy.
- Theorem. PD implies that every projective set is Lebesgue measurable, has the Baire property, and has the PSP.

A Case Study in the Extrinsic Program
--A Proof [?]of PD--

- "A proof of projective determinacy"(Martin and Steel 1989)
- What they prove is that PD holds if there are infinitely many Woodin cardinals with a measurable cardinal above them.
- Success for the Extrinsic Program [?]
  But is PD true?

What Hope for the Extrinsic Program to settle CH?

- Theorem (Levy and Solovay 1967): CH is consistent with and independent of all (“small” and “large”) LCAs that have been considered to date, provided they are consistent with ZF.
- Proof. By Cohen’s method of forcing.

What Prospects for CH?
Woodin's Program: Changing the Logic

- Ω-logic: A new logic that cannot be altered by forcing, thus avoiding the Levy-Solovay Thm.
- Woodin's Strong Ω-conjecture implies that not-CH is an Ω-consequence of ZFC + (an Ω-complete axiom).
- The proof assumes the existence of a proper class of Woodin cardinals.
- But would that prove that CH is false?

The Millennium Prize List

- The Millennium Prize List: 7 famous unsolved problems, including the Riemann Hypothesis, Poincaré Conjecture, P vs NP, etc.
- The prize: $1,000,000 each.
- The criteria for the problems on the list: Should be historic, central, important, and difficult.

The Millennium Prize List (cont'd)

- CH a prima facie candidate. Was it considered for the list? (No published rationale for exclusion of famous problems.)
- A new situation: Pereleman solved the Poincaré Conjecture but declined the prize, thus freeing up $1,000,000.
- A possible scenario: one new problem is to be added to the list; expert advice is solicited anew on its choice.

Millennium Discussion

- Should the Board add CH to the list? Usual idea of mathematical truth in its ordinary sense is no longer operative in these research programs.
- Even if experts in set theory find assumptions like a class of Woodin cardinals compelling, likelihood of their being accepted by the mathematical community at large is practically nil.

Is CH a Definite Mathematical Problem?

- My conjecture: No; in fact it is essentially indefinite (“inherently vague”).
- That is, the concepts of arbitrary set and function as used in its formulation even at the level of $\mathcal{P}(N)$ are essentially indefinite.
- This comes from my general anti-platonistic view of the nature of mathematics: it is humanly based and deals with more or less clear conceptions of mathematical structures, beginning with N.

Conceptions of Sets

- Sets are supposed to be definite totalities, determined solely by which objects are in the membership relation (∈) to them, and independently of how they may be defined, if at all.
- A is a definite totality iff the logical operation of quantifying over A, $(\forall x \in A)\ P(x)$, has a determinate truth value for each definite property $P(x)$ of elements of A.

The Structure of "all" Sets

- $(V, \in)$, where V is the universe of "all" sets.
- V itself is not a definite totality, so unbounded quantification over V is not justified on this conception. Indeed, it is essentially indefinite.
- If the operation $\mathcal{P}(\ .\ )$ is conceived to lead from sets to sets, that justifies the Power Set Axiom (Pow).

The Status of CH

- But--I believe--the assumption of $\mathcal{P}$(N), $\mathcal{P}(\mathcal{P}$(N)) as definite totalities is philosophically justified only on platonistic grounds.
- From my point of view, the conception of the totality of arbitrary subsets of any given infinite set is essentially indefinite (or inherently vague).
- For, any effort to make it definite violates the idea of what it is supposed to be about.

Is there an intermediate position?

- The concept of the continuum $\mathcal{P}(N)$ in its guise as 2^N is particularly intuitive.
- Suppose we grant the idea of 2^N or $\mathcal{P}(N)$ as a working apparently robust idea, but nothing higher in the cumulative hierarchy.
- That justifies Dedekind completeness of R w.r.t. all sets definable in 2nd order number theory.
- But CH requires for its formulation as a definite statement, $\mathcal{P}(\mathcal{P}(N))$ as a definite totality.

How can CH *not* have a definite mathematical meaning?

- There is no disputing that CH is a definite statement in the language of set theory, whether considered formally or informally; it just concerns $\mathcal{P}(\mathcal{P}(\mathrm{N}))$.
- And there is no doubt that that language involves concepts that have become an established, robust part of mathematical practice.
- But that may be because mathematical practice uses relatively little from those concepts.

A Formal Distinction Between Definite and Indefinite Concepts

- Proposal: What's definite is the domain of classical logic, what's not is that of intuitionistic logic.
- Semi-constructive systems.
- In the case of predicativity, consider systems in which quantification over natural numbers is governed by classical logic, while quantification over sets of natural numbers (and sets more generally) is governed by intuitionistic logic.

A Formal Distinction (Continued)

- In the case of set theory, where every set is conceived to be a definite totality, but the universe of sets is an indefinite totality, accept classical logic for bounded quantification while use intuitionistic logic for unbounded quantification.
- We say that a sentence A is formally definite in one of our semi-constructive systems if $A \vee \neg A$ is provable there.
- Conjecture: In a suitable semi-constructive formulation of set theory, CH is not definite.

Martin 1976

"Those who argue that the concept of set is not sufficiently clear to fix the truth-value of CH have a position which is at present difficult to assail. As long as no new axiom is found which decides CH, their case will continue to grow stronger, and our assertion that the meaning of CH is clear will sound more and more empty."

Reference

“Is the continuum hypothesis a definite mathematical problem?” http://math.stanford.edu/~feferman/papers/IsCHdefinite.pdf

The End